Universality of the Crossing Probability for the Potts Model for $q = 1, 2, 3, 4$

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The universality of the crossing probability π_{hs} of a system to percolate only in the horizontal direction was investigated numerically by a cluster Monte-Carlo algorithm for the q-state Potts model for $q = 2, 3, 4$ and for percolation $q = 1$. We check the percolation through Fortuin-Kasteleyn clusters near the critical point on the square lattice by using representation of the Potts model as the correlated site-bond percolation model. It was shown that probability of a system to percolate only in the horizontal direction π_{hs} has universal form $\pi_{hs} = A(q)Q(z)$ for $q = 1, 2, 3, 4$ as a function of the scaling variable $z = \left[b(q)L^{\frac{1}{\nu(q)}}(p - p_c(q, L))\right]^{\zeta(q)}$. Here, $p = 1 - \exp(-\beta)$ is the probability of a bond to be closed, $A(q)$ is the nonuniversal crossing amplitude, $b(q)$ is the nonuniversal metric factor, $\nu(q)$ is the correlation length index, $\zeta(q)$ is the additional scaling index. The universal function $Q(x) \simeq \exp(-|z|)$. Nonuniversal scaling factors were found numerically.

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I. INTRODUCTION

The concept of the universality and scaling relations [1] is general concept of the modern phase transition theory. The main point of the scaling theory is that in the vicinity of the critical point for a system of linear size L, the critical behavior of thermodynamical quantities can be expressed as a universal function of two variables: reduced temperature $\tau = \frac{T-T_c}{T_c}$ and external field h. The finite-size scaling of thermodynamical functions of spin models was studied theoretically and numerically [2, 3, 4, 5]. In Ref. [6], Privman and Fisher proposed the idea of the universal finite-size scaling with nonuniversal metric factors. For example, for the free-energy density of system size L and dimension d

$$f(T, h; L) = L^{-d}Y\left(C_1\tau L^{\frac{1}{\nu}}, C_2hL^{(\beta+\gamma)/\nu}\right) \qquad (1)$$

where ν is the scaling index for correlation length, β is the scaling index of magnetization and γ is the scaling index of magnetic susceptibility. For systems of different boundary conditions, aspect ratios and geometries (square, honeycomb, triangle), the scaling function $Y(x, y)$ is universal and only metric factors C_1, C_2 are system dependent. Scaling properties of thermodynamical functions of the Potts model were investigated in Refs. [7, 8].

Langlands, Pichet, Pouliot and Saint-Aubin [9] show that for site and bond percolation on square, honeycomb and triangle lattices with the aspect ratios a, $a\sqrt{3}$ and $a\sqrt{3}/2$ respectively, the crossing probability π_h of a system to percolate in the horizontal direction is the universal function of a. Hu, Lin show that by choosing a very small number of nonuniversal metric factors, all scaled data for percolation functions and the number of percolating clusters on square, honeycomb and triangle lattices may fall on the same universal scaling functions [10, 11]. Their scaling argument was $x = (p - p_c)L^{\frac{1}{\nu}}$ where p_c is the critical point, L is the lattice size, ν is the correlation length index. The scaling of crossing probabilities for the three-dimensional percolation was investigated in Ref. [12].

The continuum limit of the crossing probability $\pi_h(p_c)$ was investigated by J. Cardy by conformal field methods [13, 14, 15]. The analogous formula for the crossing probability $\pi_{hv} = \pi_h - \pi_{hs}$ was found by Watts [16]. The works of Smirnov [17, 18] analytically proved that the crossing formula holds for the continuum limit of site percolation on the triangle lattice [17, 18].

The q-state Potts model can be represented as the correlated site-bond percolation in terms of Fortuin-Kasteleyn (FK) clusters [19]. At the critical point of the second order phase transition Potts model, FK-clusters exhibit the percolation transition. So there is an intrinsic relationship between critical properties of the Potts model and percolation properties of FK clusters. The universality of the crossing probability for the Ising model on rectangle lattices of square, honeycomb and triangle geometries was investigated by Langlands *et all.* [20, 21]. Hu, Chen and Lin show the universality of the crossing probability π_h and a number of percolation clusters for the correlated site-bond percolation $q = 2$ [22].

In this paper the the crossing probabilities of FK clusters is studied numerically. We investigate the universality of the crossing probability with respect to a number of spin states q of the Potts model. We show numerically that the probability of a system to percolate only in horizontal direction π_{hs} is an universal function of the scaling variable $z = \left[b(q)L^{\frac{1}{\nu}}(p - p_c)\right]^{\zeta(q)}$ for $q = 1, 2, 3, 4$ where $p = 1 - \exp(-\beta)$ is the probability of bond to be closed, $\beta = 1/Tk_B$ is an inverse temperature, $p_c = 1 - \exp(-\beta_c)$ is the location of critical point in the p scale, $b(q)$ is a nonuniversal metric factor and $\zeta(q)$ is a scaling index, described dependence of the form of the crossing proba-

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bility on q.

We show that for each value $q = 1, 2, 3, 4$ on the square, lattice the index ζ is practically does not depend on the lattice size. Therefore, by introducing this index $\zeta(q)$, we can lay on the same curve the points for different q in the critical region.

II. CROSSING PROBABILITIES FOR THE POTTS MODEL

In the Potts model, each spin σ_i can take one of the q different values $1, \ldots, q$ and the Hamiltonian is written as $\mathcal{H} = -J \sum\limits_{(i,j)} \delta(\sigma_i, \sigma_j)$, where J is the ferromagnetic coupling constant, which we set it equal 1. The partition function of the Potts model [23] is

$$Z = \sum_{\sigma} \exp(-\beta \mathcal{H}(\omega)) = \sum_{\sigma} \prod_{(i,j)} [(1-p) + p\delta(\sigma_i, \sigma_j)] \tag{2}$$

where $\beta = 1/Tk_B$ is the inverse temperature, $p = 1 - \exp(-\beta)$ is the probability of bond to be closed and $1 - p = \exp(-\beta)$ is the probability of bond to be open; summation is performed over all spin configurations σ. Sometimes the term in square brackets is expressed via $v = \exp(\beta) - 1$: $[(1-p) + p\delta(\sigma_i, \sigma_j)] = \exp(-\beta) [1 + v\delta(\sigma_i, \sigma_j)]$, but we write Eq (2) via p to emphasize the fact that in correlated site-bond percolation [19], the probability of bond to be closed is p and to be open is $1 - p$.

For the square lattice of linear size L with periodic boundary conditions the total number of bonds is $N = 2L^2$. Let us define by $\mathcal{L}$ the graph of all the edges on the lattice. The product over all bonds (i, j) consists of N terms, so the product may be expanded as sum of 2^N terms. Let us associate to each of these 2^N terms subgraph G of graph $\mathcal{L}$, by following rule. Each of 2^N terms can be considered as the product of N factors. Each factor $[(1-p) + p\delta(\sigma_i, \sigma_j)]$ corresponds to some edge (i, j) of the graph $\mathcal{L}$. To construct G, we perform the following procedure. If this factor for edge (i, j) is equal to $1 - p$, we delete edge (i, j) from subgraph. If this factor for edge (i, j) is equal to p, we leave this edge in the subgraph. So we associate to each term in sum (2) subgraph G. Each subgraph G consists of $b(G)$ edges (closed bonds) and $C(G)$ connected components. The term in (2) corresponding to G contains factor $(1-p)^{N-b(G)} p^{b(G)}$, and delta-functions guarantee equivalence of spins in each connected component. As a result of the summation over all possible spin configurations, the contribution of this subgraph G into the partition function is $q^{C(G)}(1-p)^{N-b(G)} p^{b(G)}$. Therefore, we can replace the summation over all spin configurations in Eq. (2) by summation over all possible subgraph G on $\mathcal{L}$

$$Z = \sum_{G \in \mathcal{L}} q^{C(G)} (1-p)^{N-b(G)} p^{b(G)} \tag{3}$$

We shall keep in mind that this definition is valid even for the non-integer value of q. The partition function of the Potts model for $q \to 1$ corresponds to the percolation, where p is a probability of bond to be occupied.

Now we introduce the crossing probability π_{hs}, the probability of a system to percolate only in the horizontal direction while the percolation in the vertical direction is absent. We must distinguish it from the probability of a system to percolate in the horizontal direction irrespective to percolation in the vertical direction π_h. We define the indicator function $I_{hs}(G)$ for each subgraph G in accordance with the rule

$$I_{hs}(G) = \begin{cases} 1 & \text{if } G \text{ percolates only in horizontal direction} \\ 0 & \text{in all others cases} \end{cases} \tag{4}$$

We mean that G percolates only in the horizontal direction if it contains at least one connected component touching left and right sides of the lattice, and there are no components joining top and bottom sides. Therefore, the crossing probability $\pi_{hs}(\beta; L, q)$ of a system of size L with q possible spin states to percolate only in the horizontal direction at β can be written as

$$\pi_{hs}(\beta; L, q) = \frac{1}{Z} \sum_{G \in \mathcal{L}} I_{hs}(G) q^{C(G)} (1-p)^{N-b(G)} p^{b(G)} \tag{5}$$

Here, $p = 1 - \exp(-\beta)$. We can introduce the crossing probability for vertical direction π_{vs} by the same way. But on the square lattice $\pi_{hs} = \pi_{vs}$ and later on, investigate only π_{hs}.

Let us notice that $\pi_{hs}(\beta; L, q)$ has maximum near the critical point $\beta_c(L, q)$ because in the ordering phase $\beta > \beta_c(L, q)$, the most probable configurations contain large clusters touching top and bottom sides of the lattice, and do not contribute into π_{hs}.

III. NUMERICAL RESULTS

We use the Wolff cluster algorithm [24] to generate the different spin configurations. For each spin configuration we generate a bond configuration in accordance with [19]. Then, we break the lattice into independent clusters of connected sites by using the Hoshen-Kopelman [25] algorithm. Then, we analyze crossing properties $I_{hs}(G)$ of this configuration. Between checking the crossing we skip 5 Monte-Carlo steps. For each value of β, the averaging is performed over 10 series each of 10^5 configurations. The total number of configurations is 10^6. Sets of configurations are used for the estimation of the numerical inaccuracy. Quantity I_{hs} is an indicator function. It means that it takes values 0 or 1 for each configuration. Therefore, the resolution of our computations for π_{hs} is 10^{-6}. We compute data for $q = 1(\text{percolation}), 2, 3, 4$ and lattice sizes $L = 16, 32, 48, 64, 80, 96, 112, 128$. For each pair (q, L), we perform computation for 200 values of β (or p for percolation) in the critical region.

There is a very important question, how the scaling variable should be chosen?

For the percolation $q = 1$ the scaling variable is defined as the deviation of the bond concentration from critical point $r = p - p_c$. For the Potts model, we can take

1. $t = \frac{T - T_c}{T_c}$
2. $\tau = \frac{\beta - \beta_c}{\beta_c}$
3. $r = (1 - \exp(-\beta)) - (1 - \exp(-\beta_c) = p - p_c$

The first variant $t = \frac{T - T_c}{T_c}$ is usual for the investigation of thermodynamic quantities [1, 23], the second one is widely used for the approximation of magnetic susceptibility χ of the Ising model in the critical region [26], and the third variant $r = p - p_c$ may be argued by the fact that for the site-bond correlated percolation, the probability of bond to be occupied is $p = 1 - \exp(-\beta)$ – see equations (2), (3).

Let us evaluate the symmetry of the crossing probability as a function of all these variables by comparing third moments of the function π_{hs}. We assume that π_{hs} must be the most symmetric function as a function of "right" variable. We calculate four moments of $\pi_{hs}(x)$ as a functions of $x = t, \tau, r$ by numerical integration in accordance with formula $\mu_0 = \int \pi_{hs}(x)dx$, $\mu_1 = \frac{1}{\mu_0}\int x\pi_{hs}(x)dx$, $\mu_k = \frac{1}{\mu_0}\int (x - \mu_1)^k \pi_{hs}(x)dx$. Results of computations μ_3 for the lattice size $L = 128$ are given in Table I. We

TABLE I: The third moment μ_3 of the crossing probability π_{hs}, computed for different variables t, τ and r

q	2	3	4
t	$-7.1(3) \times 10^{-8}$	$-9.1(2) \times 10^{-9}$	$-8(1) \times 10^{-10}$
τ	$-2.1(6) \times 10^{-8}$	$-2.3(2) \times 10^{-9}$	$-2(77) \times 10^{-12}$
r	$2.5(2) \times 10^{-9}$	$2.8(1) \times 10^{-11}$	$1.3(5) \times 10^{-11}$

see that for all the cases μ_3 is smaller for scaling variable r, with one exception $q = 4$, τ, when numerical error is approximately thirty times greater than value $\mu_3(\tau)$ and six times greater than $\mu_3(r)$. For all other lattice sizes $L = 16, \ldots, 112$ the third moment μ_3 calculated by using variable r, is smaller than for t or for τ. Therefore, we will work with the scaling variable $p = 1 - \exp(-\beta)$, the probability of bond to be closed for $q = 2, 3, 4$. The critical point of the q-state Potts model on the infinite lattice is [23] $p_c(q) = \frac{\sqrt{q}}{\sqrt{q}+1}$. The bond percolation critical point $p_c(q = 1) = \frac{1}{2}$ can be obtained from this formula. Variable $p = 1 - \exp(-\beta)$ naturally provides crossover from the percolation to the Potts model. We can plot the crossing probability π_{hs} as a function of p for the percolation and the Potts model on the same graph.

In Fig.1, data for $\pi_{hs}(p; L, q)$ are plotted for $q = 1, 2, 3, 4$ and $L = 32, 64, 128$. We can see shift of the critical point $p_c(q, L)$ for finite lattice sizes L. We also

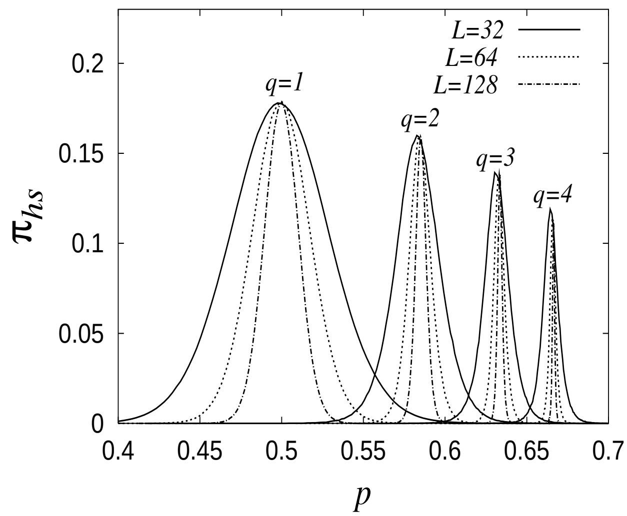

FIG. 1: The crossing probability $\pi_{hs}(p; L, q)$ for $q = 1, 2, 3, 4$ and $L = 32, 64, 128$

see the change of the width of the function due to size scaling. Let us try to identify the shape of π_{hs} by analyzing the ratio $\frac{\mu_4}{\mu_2^2}$. We know that at least for the percolation, function π_{hs} looks like the Gaussian function [27] $\sim A\exp(-x^2)$, and for the Gaussian function we expect $\frac{\mu_4}{\mu_2^2}(Gaussian) = 3$. But, in reality, crossing probabilities are not Gaussian. The moments ratio for function $\pi'_h(p)$ (the derivative of crossing probability $\pi_h(p)$ with respect to p), for the percolation model was found by Ziff $\frac{\mu_4}{\mu_2^2}(q = 1) \simeq 3.20(5)$ [28] and more precisely $\frac{\mu_4}{\mu_2^2}(q = 1) \simeq 3.176(4)$ [29] and by Hovy and Aharony $\frac{\mu_4}{\mu_2^2}(q = 1) \simeq 3.174(25)$ [30]. The explanation of the non-Gaussian crossing probability shape was first given by Berlyard and Wehr [31] and was also discussed in Ref. [32]. Newman and Ziff verify that the tails of the crossing probability falls of as $\exp(-c(p - p_c)^{4/3})$, therefore the tail of the distribution is characterized by the correlation length exponent [33, 34]. However, as the first approximation we introduce an additional scaling index ζ and check moment ratios for function $g(x; \zeta) = A\exp(-x^\zeta)$. For this function $g(x; \zeta)$ the moment ratios are

$$\frac{\mu_4}{\mu_2^2}(\zeta) = \frac{\Gamma(\frac{1}{\zeta})\Gamma(\frac{5}{\zeta})}{\Gamma(\frac{3}{\zeta})^2} \tag{6}$$

We calculate moments for several values of ζ and put these data into the first (ζ) and second ($\frac{\mu_4}{\mu_2^2}(\zeta)$) rows of the Table II. The choice of values ζ will be argued later. Results of computation of the moments ratio $\frac{\mu_4}{\mu_2^2}(q)$ for the crossing probability $\pi_{hs}(p; L, q)$ for $L = 32, 128$ is placed in the fourth and fifth row of the Table II. We see that the moments ratio practically does not depend upon the lattice size L. We check this fact for others lat-

tice sizes. Moment ratios for $g(x;\zeta)$ slightly differ from

TABLE II: The ratio of moments $\frac{\mu_4}{\mu_2^2}$ for $g(x,\zeta)$ and $\pi_{hs}(p;L,q)$

ζ	2	3/2	4/3	6/5
$\frac{\mu_4}{\mu_2^2}(\zeta)$ analytically	3	3.76195	4.22219	4.7434
q	1	2	3	4
$\frac{\mu_4}{\mu_2^2}(q)$ numerically $L=32$	3.145(6)	3.871(7)	4.577(14)	5.28(2)
$\frac{\mu_4}{\mu_2^2}(q)$ numerically $L=128$	3.18(2)	3.91(8)	4.56(2)	5.30(4)

moment ratios for π_{hs} (this fact will be explained later), but we can try to approximate the crossing probability by the function $g(x,\zeta)$ and then compare results of approximation with numerical data.

Below, we describe the fitting procedure. As we can see from Fig.1, there are many nonuniversal scaling factors for π_{hs}: the amplitude of the crossing probability $A(L,q)$, the finite size critical point $p_c(L,q)$, which differs from $p_c(L=\infty,q)=\frac{\sqrt{q}}{\sqrt{q}+1}$, the nonuniversal scaling factor $B(L,q)$, which provide the finite size scaling of the function, and the additional scaling index $\zeta(L,q)$.

We perform the four-parametric fit of $\pi_{hs}(p;L,q)$ by the function (7)

$$F(p;L,q)=A(q,L)\exp(-\left[B(L,q)(p-p_c(L,q))\right]^{\zeta(L,q)}) \quad (7)$$

We use points $\log(\pi_{hs})>-9$ for this fit and the log scale for the ordinate axis. As a result of the approximation we get a set of scaling amplitudes $A(q,L)$, nonuniversal metric $B(q,L)$, critical points $p_c(q,L)$ and scaling indices $\zeta(q,L)$. Then, we use this scaling factors to adjust our numerical data onto one line for $q=1,2,3,4$ and $L=32,128$. We plot $f=|\log(\pi_{hs}(p;L,q))-\log(A(L,q))|$ as a function of the new scaling variable $z=\left[B(L,q)\left(p-p_c(L,q)\right)\right]^{\zeta(L,q)}$ in Fig.2.

The scaling straight lines $|z|$ shown in Fig.2, correspond to the fitting function $F(z)=\exp(-|z|)$. We see that all points lie on the one curve, and this curve is very close to $|z|$ in the range $0.1<z<3$. So, we might expect that our fitting procedure is correct. However, on the graph, the tail points deviate from the line $|z|$, and this deviation explains the fact that moment ratios for $\pi_{hs}(p;L,q)$ in the Table II differ from analytical values for $g(x,\zeta)$. Results of approximation for the crossing amplitude $A(L,q)$ are placed in Table III. For each value of $q=1,2,3$ amplitudes $A(L,q)$ are equal within our numerical accuracy of the approximation. Therefore, we can conclude that the scaling amplitude depends upon q and depends weakly upon L.

Results of the approximation for the scaling index $\zeta(L,q)$ are represented in the Table IV. We can see that this scaling index practically does not depend upon lattice size. We assume that for each value q we have the additional scaling index $\zeta(q)$, which is nondependent upon the lattice size. Test values of ζ in the first row of Table II are chosen to be close to $\zeta(q)$. It seems, that $\zeta(3)=4/3$, $\zeta(4)=6/5$.

We know that in accordance with the scaling theory for each fixed value q the crossing probability must be a universal function of variable $x=L^{\frac{1}{\nu(q)}}(p-p_c(L,q))$. Therefore, we approximate the numerical data for $B(L,q)$ (see Fig.3) by function $b(q)L^{y(q)}$ and represent results in Table V. Results of the approximation

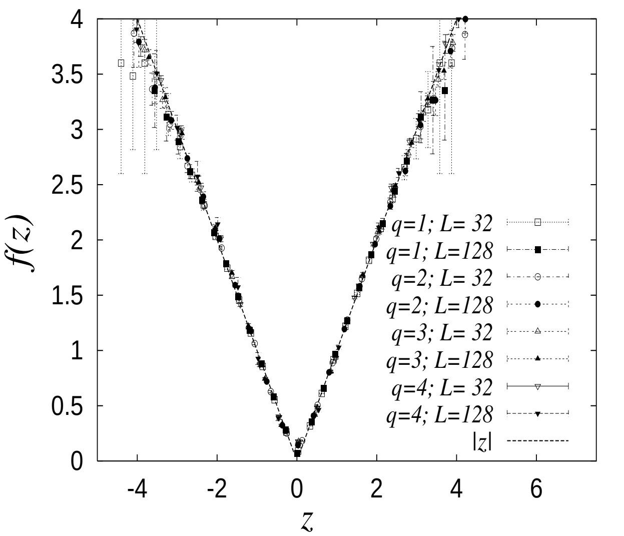

FIG. 2: The scaled crossing probability $f(z)=|\log(Q(z))|=|\log(\pi_{hs})-\log(A(L,q))|$ as a function of the new scaling variable $z=[B(L,q)(p-p_c(L,q))]^{\zeta(L,q)}$ for $q=1,2,3,4$ and $L=32,128$.

TABLE III: The scaling amplitude $A(L,q)$ for $L=16,32,64,128$

q	1	2	3	4
$A(L=16)$	0.1806(2)	0.166(1)	0.151(2)	0.137(2)
$A(L=32)$	0.1794(3)	0.167(1)	0.151(2)	0.132(2)
$A(L=64)$	0.1806(3)	0.167(1)	0.148(1)	0.128(2)
$A(L=128)$	0.1794(3)	0.166(1)	0.149(2)	0.122(2)

TABLE IV: The scaling index $\zeta(L,q)$ for $L=16,32,64,128$

q	1	2	3	4
$\zeta(L=16)$	1.921(3)	1.577(12)	1.398(15)	1.264(18)
$\zeta(L=32)$	1.900(4)	1.557(12)	1.364(13)	1.238(14)
$\zeta(L=64)$	1.886(4)	1.55(12)	1.367(13)	1.218(15)
$\zeta(L=128)$	1.887(5)	1.545(14)	1.337(13)	1.198(14)

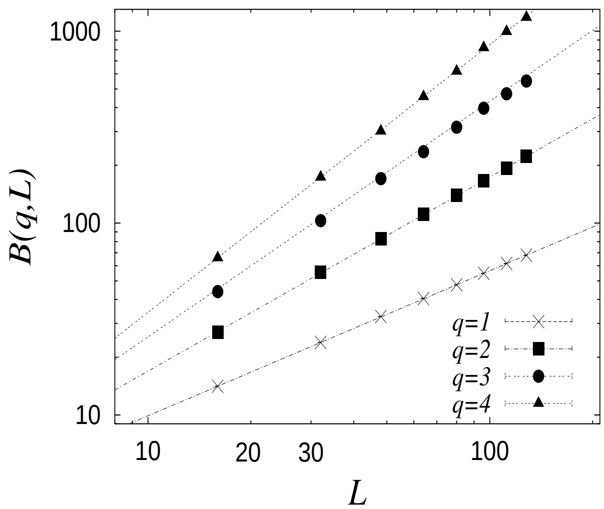

FIG. 3: Nonuniversal metric factors $B(L,q)$. Results of approximation by the function $b(q)L^{y(q)}$ are shown by lines (see Table V).

$b(q)L^{y(q)}$ are also shown in Fig.3 by lines. We can see

TABLE V: The approximation of the nonuniversal metric factors $B(L,q)$ by the function $b(q)L^{y(q)}$. The inverse correlation length index $\frac{1}{\nu(q)}$ is added for the comparison.

q	1	2	3	4
$b(q)$	1.737(6)	1.65(3)	1.51(3)	1.38(3)
$y(q)$	0.756(8)	1.011(3)	1.218(16)	1.39(6)
$\frac{1}{\nu(q)}$	0.75	1	1.2	1.5

that the thermal scaling index $y(q)$ is very close to the analytical value of the inverse correlation length index $\frac{1}{\nu(q)}$ [23], which is represented in the Table V for the comparison. This fact once more confirms our approximation procedure. The exception is the case $q = 4$, for which there is a difference between $y(q = 4)$ and $\frac{1}{\nu(q=4)}$.

Many critical quantities in the Potts model $q = 4$ exhibit logarithmic corrections [35, 36, 37, 38]. These logarithmic corrections explain the difference between analytical value $\frac{1}{\nu(q=4)} = 1.5$ and numerical approximation for the scaling index $y = 1.39(6)$.

IV. CONCLUSIONS

To summarize results, we can conclude that the crossing probability $\pi_{hs}(p; L, q)$ is a universal function $Q(z)$

$$\pi_{hs}(p; L, q) = A(q)Q\left(\left[b(q)L^{\frac{1}{\nu(q)}}(p - p_c(L,q))\right]^{\zeta(q)}\right) \tag{8}$$

of the scaling variable $z = \left[b(q)L^{\frac{1}{\nu(q)}}(p - p_c(L,q))\right]^{\zeta(q)}$. As we can conclude from Fig.2, the function $Q(z)$ looks like the exponent $Q(z) \simeq \exp(-|z|)$ on the interval $0.1 < z < 3$, but deviates from it in the vicinity of 0 and on the tails $|z| > 3$.

Let us pay attention to the new details of this work. We consider the universality of the crossing probability for different values of q by adding the scaling index $\zeta(q)$. We work in the scale p, where $p = 1 - \exp(-\beta)$ is the probability of a bond to be closed instead of the usual scale $t = (T - T_c)/T_c$ to make the crossing probability symmetric. We find numerically the scaling index $\zeta(q)$. The universal function looks like $Q(z) \simeq \exp(-|z|)$ on the interval $0.1 < z < 3$.

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